

A Sharp Estimate for a Multilinear Singular Integral in \mathbf{R}^n

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§0. Introduction. The purpose of this paper is to prove a good λ inequality for certain multilinear singular integrals in \mathbf{R}^n . The operators considered here are the maximal and limiting operators of the singular integral

$$C_\varepsilon(\nabla A, f)(x) = \int_{|x-y|>\varepsilon} P_2(A; x, y) \frac{\Omega(x-y)}{|x-y|^{n+1}} f(y) dy$$

where $P_2(A; x, y) = A(x) - A(y) - \nabla A(y) \cdot (x - y)$ is the second order Taylor series remainder of A , Ω satisfies certain homogeneity, smoothness and symmetry conditions, $\nabla A \in \text{BMO}$ and $f \in L^p(\mathbf{R}^n)$. The maximal function $C_*(\nabla A, f)(x) = \sup_{\varepsilon>0} |C_\varepsilon(\nabla A, f)(x)|$ is shown to be controlled by a “sharp function” of ∇A and a Hardy-Littlewood maximal function of f .

The operators considered in this paper are related to the commutators studied by Calderón in [2]. An extension of Calderón’s result to \mathbf{R}^n by Baishansky and Coifman [1] is an essential part of the proof of the main result of this paper.

The methods in this paper are similar to those used by Coifman and Meyer in studying the second commutator on the line and by Coifman, Rochberg and Weiss for studying the first commutator in \mathbf{R}^n . However, the higher order Taylor series remainder occurring in the operators considered in this paper require a modification of the procedure.

In studying commutators of singular integrals, Coifman, Rochberg and Weiss [5] replace the term $b(y) - b(x)$ by $b(y) - m_{Q_x}(b)$ where $m_{Q_x}(b)$ is the average of b over the cube Q_x centered at x . They obtain an L^p estimate for this new operator in terms of the BMO norm of b which is independent of the choice of the cube Q_x . By shrinking the cube one gets the estimate for the commutator.

This procedure doesn’t work in the Taylor series remainder case. Instead we use the fact that $P_2(A; x, y) = P_2(A_Q; x, y)$ where $A_Q(x) = A(x) - \sum_{j=1}^n m_Q(D_j A)x_j$ and the fact that $(A_Q(x) - A_Q(y))/|x - y|$ can be estimated in terms of a “sharp” function of ∇A .

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§1. Statement of results. Before stating the main results, it is necessary to introduce some notation and give some definitions.

Let $g = (g_1, \dots, g_n)$ where $g_j \in L^p(\mathbf{R}^n)$. Assume Q is a cube in \mathbf{R}^n with sides parallel to the axes. Let $|Q|$ denote the Lebesgue measure of Q .

Let $m_Q(g) = (m_Q(g_1), \dots, m_Q(g_n))$ where $m_Q(g_j) = (1/|Q|) \int_Q g_j(x) dx$. Let $S_q(g)(x) = \sup_{Q \ni x} \{(1/|Q|) \int_Q |g(x) - m_Q(g)|^q dx\}^{1/q}$.

If the L^∞ norm of $S_q(g)$ is finite we say that g is a function of Bounded Mean Oscillation (BMO). We write $g \in \text{BMO}$ and define $\|g\|_{\text{BMO}_q} = \|S_q(g)\|_\infty$. A result of John and Nirenberg [6] implies the equivalence of the BMO norms for $1 \leq q < \infty$. ($\|g\|_{\text{BMO}_q}$ is in fact a norm if we identify functions that differ by a constant.)

Let $f \in L^p(\mathbf{R}^n)$, $1 < p < \infty$, and $\|\nabla A\|_{\text{BMO}_q} < \infty$ for $1 \leq q < \infty$, where $\nabla A = (D_1 A, \dots, D_n A)$ and $D_j A = \partial A / \partial x_j$. Define

$$C_\varepsilon(\nabla A, f)(x) = \int_{|x-y|>\varepsilon} \frac{A(x) - A(y) - \nabla A(y) \cdot (x-y)}{|x-y|^{n+1}} \Omega(x-y) f(y) dy$$

$$C_*(\nabla A, f)(x) = \sup_{\varepsilon>0} |C_\varepsilon(\nabla A, f)(x)|$$

where

1. Ω is homogeneous of degree zero.
2. $|\Omega(x) - \Omega(y)| \leq c|x-y|$ for $|x| = |y| = 1$.
3. $\int_{|x|=1} x_j \Omega(x) dx = 0, \quad j = 1, \dots, n$.

For $f \in L^p(\mathbf{R}^n)$, Q a cube in \mathbf{R}^n with sides parallel to the axes,

$$\Lambda_p(f)(x) = \sup_{Q \ni x} \left\{ \frac{1}{|Q|} \int_Q |f(x)|^p dx \right\}^{1/p}.$$

We can now state the main result of this paper.

Theorem (1.1). For $f \in L^p(\mathbf{R}^n)$ and $\nabla A \in \text{BMO}$, there exists $\gamma_0 > 0$ such that for $0 < \gamma < \gamma_0$ and $\lambda > 0$

$$|x : C_*(\nabla A, f)(x) > 3\lambda, S_q(\nabla A)(x) \Lambda_p(f)(x) \leq \gamma\lambda| \leq c\gamma^r |x : C_*(\nabla A, f)(x) > \lambda|$$

where $1/p + 1/q = 1/r$ and $q > n$, and $q > p'$ where $1/p + 1/p' = 1$.

Corollary (1.2). If $\nabla A \in \text{BMO}$ and $f \in L^p(\mathbf{R}^n)$, $p > n$, then the limiting operator

$$C(\nabla A, f)(x) = \lim_{\varepsilon \rightarrow 0} C_\varepsilon(\nabla A, f)(x)$$

exists almost everywhere and satisfies the estimate

$$\|C(\nabla A, f)\|_p \leq C \|\nabla A\|_{\text{BMO}} \|f\|_p.$$

§2. Some useful estimates. The proof of Theorem 1.1 requires several lemmas which are listed below. These lemmas are adaptations of standard estimates and are stated without proof. For references, Lemma 2.1 can be found in [3], Lemma 2.2 can be found in [6], and the essentials of Lemma 2.5 can be found in [5].

Lemma (2.1). Let $A(x)$ be a function defined in \mathbf{R}^n . Assume $|\nabla A| \in L^q(\mathbf{R}^n)$. Then for $q > n$

$$|A(x) - A(y)| \leq c|x - y| \left\{ \frac{1}{I_x^y} \int_{I_x^y} |\nabla A(z)|^q dz \right\}^{1/q}$$

where I_x^y is the cube centered at x , with sides parallel to the axes and whose diameter is $2n^{1/2}|x - y|$.

Lemma (2.2). Let Q be a cube with sides parallel to the axes. Then,

(2.3) if $\rho > 1$, $|m_Q(\nabla A) - m_{\rho Q}(\nabla A)| \leq c\rho^n S_q(\nabla A)(x)$ for all $x \in \rho Q$, where ρQ is the cube with the same center as Q and sides expanded by a factor of ρ .

(2.4) For $x \in Q$ and $|x - y| > 2d(Q)$, ($d(Q)$ is the diameter of Q).

$$|m_Q(\nabla A) - m_{I_x^y}(\nabla A)| \leq c \left[1 + \log \frac{|x - y|}{d(Q)} \right] S_q(\nabla A)(x_1)$$

for all $x_1 \in Q$.

Lemma (2.5). Let $f \in L^p(\mathbf{R}^n)$ and $\nabla A \in \text{BMO}$. Then

(2.6) for $|x - x_1| < \varepsilon$, $\rho > 1$,

$$\left\{ \int_{\varepsilon < |x-y| < \rho\varepsilon} \frac{|f(y)|^p}{|x - y|^n} dy \right\}^{1/p} \leq c \Lambda_\rho(f)(x_1).$$

(2.7) For $|x - x_1| < \varepsilon$,

$$\left\{ \varepsilon \int_{|x-y| > \varepsilon} \left[1 + \log \frac{|x - y|}{\varepsilon} \right] \frac{|f(y)|^p}{|x - y|^{n+1}} dy \right\}^{1/p} \leq \Lambda_\rho(f)(x_1).$$

(2.8) Let Q be a cube in \mathbf{R}^n with sides parallel to the axes. Let $x_1 \in Q$. Then if $\rho > 1$,

$$\left\{ \int_{d(Q) < |x-y| < \rho d(Q)} \frac{|\nabla A(y) - m_Q(\nabla A)|^q}{|x - y|^n} dy \right\}^{1/q} \leq c_\rho S_q(\nabla A)(x_1).$$

(2.9) If x_1 and Q are as in (2.8),

$$\left\{ d(Q) \int_{|x-y|>d(Q)} \left[1 + \log \frac{|x-y|}{d(Q)} \right] \frac{|\nabla A(y) - m_Q(\nabla A)|^q}{|x-y|^{n+1}} dy \right\}^{1/q} \leq c S_q(\nabla A)(x_1).$$

Lemma (2.10). Let Q be a cube in \mathbf{R}^n with sides parallel to the axes and let $x, x_1 \in Q$. Let $f \in L^p(\mathbf{R}^n)$ and $\nabla A \in \text{BMO}$. Then for $q \geq p'$ where $1/p + 1/p' = 1$,

$$(2.11) \int_{d(Q) < |x-y| < \rho d(Q)} \frac{|\nabla A(y) - m_Q(\nabla A)||f(y)|}{|x-y|^n} dy \leq c_\rho S_q(\nabla A)(x_1) \Lambda_p(f)(x_1)$$

and

$$(2.12) \quad d(Q) \int_{|x-y|>d(Q)} \left[1 + \log \frac{|x-y|}{d(Q)} \right] \frac{|\nabla A(y) - m_Q(\nabla A)||f(y)|}{|x-y|^{n+1}} dy \leq c_\rho S_q(\nabla A)(x_1) \Lambda_p(f)(x_1).$$

Proof. These estimates follow from applying Hölder’s inequality to the integrals and using Lemma (2.5).

Lemma (2.13). Let A be a function defined in \mathbf{R}^n . Assume $\nabla A \in \text{BMO}$. Let Q be a cube in \mathbf{R}^n with sides parallel to the axes. Set

$$A_Q(x) = A(x) - \sum_{j=1}^n m_Q(D_j A) x_j. \text{ Then,}$$

$$(2.14) \quad \nabla A_Q(y) = \nabla A(y) - m_Q(\nabla A) \text{ and } C_\epsilon(\nabla A_Q, f)(x) = C_\epsilon(\nabla A, f)(x)$$

(2.15) For $x \in Q, |x - y| > 2d(Q)$, and $q > n$

$$|A_Q(x) - A_Q(y)| \leq c|x - y| \left[1 + \log \frac{|x - y|}{d(Q)} \right] S_q(\nabla A)(x_1)$$

for any $x_1 \in Q$.

(2.16) For $x, x_0 \in Q$ and $q > n$,

$$|A_Q(x) - A_Q(x_0)| \leq c|x - x_0| \left[\frac{d(Q)}{|x - x_0|} \right]^n S_q(\nabla A)(x_1)$$

for any $x_1 \in Q$.

Remark (2.17). Let $\Omega(x)$ be defined in \mathbf{R}^n , homogeneous of degree zero and satisfy the condition $|\Omega(x) - \Omega(y)| \leq c|x - y|$ for $|x| = |y| = 1$. Assume Q is a cube in \mathbf{R}^n with sides parallel to the axes. Then if $x, x_0 \in Q$ and $|x - y| > 2d(Q)$,

$$\left| \frac{\Omega(x-y)}{|x-y|^{n+1}} - \frac{\Omega(x_0-y)}{|x_0-y|^{n+1}} \right| \leq c \frac{|x-x_0|}{|x_0-y|^{n+2}}.$$

§3. Proof of Theorem (1.1). The basic idea of the proof is to write $f(x)$ as the sum of two functions, one supported on a compact set and the other supported away from the singularity of the operator $C(\nabla A, f)(x)$. Established results allow us to make a weak type estimate for the function with compact support. The methods used in a paper of Coifman, Rochberg and Weiss [5] enable us to make a pointwise estimate for the function supported away from the singularity.

Let $X = \{x : C_*(\nabla A, f)(x) > \lambda\}$ and let X^c denote its complement. Using a Whitney decomposition (see Chapter VI of [7]), $X = \bigcup Q_j$ where the Q_j 's are parallel to the axes, the Q_j 's have pairwise disjoint interiors and the distance from the cubes Q_j to the complement of X is comparable to their diameters. Specifically

$$(3.1) \quad d(Q_j) \leq \text{dist}(Q_j, X^c) \leq 4d(Q_j).$$

Remark (3.2). A consequence of (3.1) is that for any j , the cube $10n^{1/2} Q_j$ intersects the complement of X . Hence, there is a point $x_0 \in 10n^{1/2} Q_j$ such that $C_*(\nabla A, f)(x_0) \leq \lambda$.

To prove Theorem (1.1) it suffices to prove it for each Q_j . The disjointness of the Q_j 's then allows us to get Proposition (1.1) by summing on j .

Choose a cube Q_j containing a point z where $S_q(\nabla A)(z) \Lambda_p(f)(z) \leq \gamma\lambda$. (If no such point exists, the desired estimate is immediately satisfied.)

Let

$$\begin{aligned} Q &= 10\sqrt{n} Q_j \\ \bar{Q} &= 5\sqrt{n} Q = 50n Q_j \\ \bar{\bar{Q}} &= 4n \bar{Q} = 200n^2 Q_j. \end{aligned}$$

Let $f_1 = f_{\chi_{\bar{Q}}}$ where $\chi_{\bar{Q}}$ is the characteristic function of the cube \bar{Q} . Let $f_2 = f - f_1$. Then f_1 and f_2 are the functions mentioned at the beginning of the proof and it remains to make the appropriate estimate on each.

The f_1 Estimate: $|x : C_*(\nabla A, f_1)(x) > \beta\lambda| \leq c(\gamma/\beta)^r |\bar{\bar{Q}}|$.

If $|\nabla A| \in L^q(\mathbf{R}^n)$, $f \in L^p(\mathbf{R}^n)$ and Ω satisfies the hypothesis of Theorem (1.1), then we know from previous results that $\|C_*(\nabla A, f)\|_r \leq c\|\nabla A\|_q \|f\|_p$ where $1/p + 1/q = 1/r < 1$. Specifically, we can write $C = C^1 + C^2$ where

$$\begin{aligned} C^1(\nabla A, f)(x) &= \text{p.v.} \int_{\mathbf{R}^n} \frac{A(x) - A(y)}{|x-y|^{n+1}} \Omega(x-y) f(y) dy \\ C^2(\nabla A, f)(x) &= \text{p.v.} \int_{\mathbf{R}^n} \frac{\nabla A(y) \circ (x-y)}{|x-y|^{n+1}} \Omega(x-y) f(y) dy. \end{aligned}$$

Let C_*^1 and C_*^2 denote the corresponding maximal functions. A paper of Coifman and Baishansky [1] gives the strong type L^p estimate for C_*^1 and standard Calderón-Zygmund results give the strong type estimates for C_*^2 . The strong type estimates for C_* imply the weak type estimate that for $|\nabla A| \in L^q(\mathbb{R}^n), f \in L^p(\mathbb{R}^n)$ and $1/p + 1/q = 1/r < 1$

$$(3.3) \quad |x : C_*(\nabla A, f)(x) > \lambda| \leq c \left[\frac{\|\nabla A\|_q \|f\|_p}{\lambda} \right]^r.$$

We want to show that $C_*(\nabla A, f_1)(x)$ satisfies the same weak type estimate for $x \in Q$ but $|\nabla A|$ is not necessarily in L^q so we can't apply (3.3) directly. Furthermore, we can't simply multiply A by the characteristic function of \bar{Q} since this would destroy the differentiability of A . We proceed more carefully.

Remark (3.4). There exists for each cube Q_j , a function $\phi_j \in C_0^\infty$ such that $\phi_j(x) \equiv 1$ for $x \in \bar{Q}$, $\phi_j(x) \equiv 0$ for $x \notin \bar{Q}$ and $|\phi(x)| \leq 1$ for all x . In addition, the gradient of ϕ satisfies:

$$(3.5) \quad |\nabla \phi_j(x)| \leq \frac{c_0}{d(\bar{Q})}$$

and the constant c_0 in (3.5) does not depend on the choice of cube Q_j . (The independence of the constant c_0 enables us to drop the subscript j in (3.4) and henceforth we will refer to the function as ϕ .) See page 170 of [7] for details.

Let x_0 be a point on the boundary of \bar{Q} . Let $A_\phi(y) = \phi(y)[A_Q(y) - A_Q(x_0)]$ where we recall that $A_Q(x) = A(x) = -\sum_{j=1}^n m_Q(D_j A)x_j$.

To establish the appropriate estimate for $C_*(\nabla A, f_1)$ we use the equality

$$(3.6) \quad C_\epsilon(\nabla A_\phi, f_1)(x) = C_\epsilon(\nabla A, f_1)(x) \quad \text{for any } x \in Q$$

and the estimates

$$(3.7) \quad \|\nabla A_\phi\|_q \leq c |\bar{Q}|^{1/q} S_q(\nabla A)(x) \quad \text{for any } x \in Q$$

and

$$(3.8) \quad \|f_1\|_p \leq c |\bar{Q}|^{1/p} \Lambda_p(f)(x) \quad \text{for any } x \in Q.$$

(3.6) is obvious and (3.8) follows from the definition of Λ_p . To show (3.7)

$$\nabla A_\phi(y) = \nabla \phi(y)[A_Q(y) - A_Q(x_0)] + \phi(y)[\nabla A(y) - m_Q(\nabla A)].$$

Taking absolute values, using Lemma (2.1) and estimates (2.4) and (3.5)

$$|\nabla A_\phi(y)| \leq c |\nabla \phi(y)| |x_0 - y| \left\{ \frac{1}{|I_y^{x_0}|} \int_{I_y^{x_0}} |\nabla A(y) - m_Q(\nabla A)|^q dy \right\}^{1/q} \\ + |\nabla A(y) - m_Q(\nabla A)| \leq c [S_q(\nabla A)(x) + |\nabla A(y) - m_Q(\nabla A)|]$$

for any $x \in Q$. Using the fact that ∇A_ϕ is supported in \bar{Q} , we get the L^q estimate

$$\|\nabla A_\phi\|_q \leq c |\bar{Q}|^{1/q} S_q(\nabla A)(x)$$

for any $x \in Q$.

Using (3.6), (3.7), (3.8) and the assumption that there is a $z \in Q_j$ such that $S_q(\nabla A)(z)\Lambda_p(f)(z) \leq \gamma\lambda$, we can apply the weak type estimate (3.3) to get (for $1/p + 1/q = 1/r < 1$),

$$\begin{aligned} |x : C_*(\nabla A_\phi, f_1)(x) > \beta\lambda| &\leq c \left[\frac{\|\nabla A_\phi\|_q \|f_1\|_p}{\beta\lambda} \right]^r \\ &\leq c \left[\frac{|\bar{Q}|^{1/q} S_q(\nabla A)(x) |\bar{Q}|^{1/p} \Lambda_p(f)(x)}{\beta\lambda} \right]^r \\ &\leq c \left(\frac{\gamma}{\beta} \right)^r |\bar{Q}|. \end{aligned}$$

The f_2 estimate: $d(Q) \leq \varepsilon \leq 1000n d(Q)$.

The estimate for $C_\varepsilon(\nabla A, f_2)(x)$ depends on how large ε is in comparison to $d(Q)$. There are three cases: $\varepsilon < d(Q)$ which is ignored since then $C_\varepsilon(\nabla A, f_2)(x) = C_{d(Q)}(\nabla A, f_2)(x)$; $d(Q) \leq \varepsilon \leq 1000n d(Q)$ and $\varepsilon > 1000n d(Q)$. For the moment we assume $d(Q) \leq \varepsilon \leq 1000n d(Q)$.

Let $A_Q(x) = A(x) - \sum_{j=1}^n m_Q(D_j A)x_j$. By (2.14)

$$C_\varepsilon(\nabla A, f)(x) = \int_{|x-y|>\varepsilon} k(x,y)f(y) dy$$

where

$$k(x,y) = \frac{A_Q(x) - A_Q(y) - A_Q(y) \cdot (x - y)}{|x - y|^{n+1}} \Omega(x - y).$$

Choose a point $x_0 \in Q \cap X^c$ and let $x \in Q$. We recall that X^c is the complement of $X = \{x : C_*(\nabla A, f)(x) > \lambda\}$ so that $C_\varepsilon(\nabla A, f)(x_0) \leq \lambda$. Adding and subtracting $C_\varepsilon(\nabla A, f_2)(x_0)$, writing $f_2 = f - f_1$ where convenient and accounting for the overlap in the regions of integration we can write

$$\begin{aligned} (3.10) \quad C_\varepsilon(\nabla A, f_2)(x) &= \int_{|x-y|>\varepsilon} [k(x,y) - k(x_0,y)] f_2(y) dy \\ &+ \int_{D_1} k(x_0,y) f_2(y) - \int_{D_2} k(x_0,y) f_2(y) dy \\ &+ \int_{|x_0-y|>\varepsilon} k(x_0,y) f(y) dy \\ &- \int_{|x_0-y|>\varepsilon} k(x_0,y) f_1(y) dy \end{aligned}$$

where $D_1 = \{y : |x - y| > \varepsilon \text{ and } |x_0 - y| < \varepsilon\}$ and $D_2 = \{y : |x_0 - y| > \varepsilon \text{ and } |x - y| \leq \varepsilon\}$.

We next note that $f_2(y) = 0$ for $d(Q) > |x_0 - y|$, $D_1 \cup D_2 \subset \{y : \varepsilon/3 < |x_0 - y| < 2\varepsilon\}$, $f_1(y) = 0$ for $|x_0 - y| > 1000n d(Q)$, $|f_1(y)| \leq |f(y)|$ and $|f_2(y)| \leq |f(y)|$.

Taking absolute values we have

$$\begin{aligned}
 (3.11) \quad \text{right hand side of (3.10)} &\leq \left| \int_{|x-y|>\varepsilon} [k(x,y) - k(x_0,y)] f_2(y) dy \right| \\
 &+ \int_{d(Q)<|x-y|<5000nd(Q)} |k(x,y)f(y)| dy \\
 &+ 2 \int_{d(Q)<|x_0-y|<5000nd(Q)} |k(x_0,y)f(y)| dy \\
 &+ \left| \int_{|x_0-y|>\varepsilon} k(x_0,y)f(y) dy \right|.
 \end{aligned}$$

The last integral is less than or equal to λ since $x_0 \notin X$. The second and third integrals are bounded by $c S_q(\nabla A)(z) \Lambda_p(f)(z) \Delta c \gamma \lambda$. This follows by first using (2.15) to estimate $k(x,y)$ and then applying (2.11) to estimate the integrals.

To estimate the first integral,

$$\begin{aligned}
 (3.12) \quad k(x,y) - k(x_0,y) &= [A_Q(x) - A_Q(y) - \nabla A_Q(y) \cdot (x - y)] \\
 &\cdot \left[\frac{\Omega(x - y)}{|x - y|^{n+1}} - \frac{\Omega(x_0 - y)}{|x_0 - y|^{n+1}} \right] \\
 &+ [A_Q(x) - A_Q(x_0)] \frac{\Omega(x_0 - y)}{|x_0 - y|^{n+1}} \\
 &+ \nabla A_Q(y) \cdot (x - x_0) \frac{\Omega(x_0 - y)}{|x_0 - y|^{n+1}}.
 \end{aligned}$$

Multiply both sides by $f_2(y)$, integrate over $|x - y| > \varepsilon$ and take absolute values. The estimate on the right is at least as big if the region of integration is $|x - y| > 2d(Q)$. Take absolute values inside the integrals, replace $|f_2|$ by $|f|$ and replace $1/|x_0 - y|$ by $1/|x - y|$. We then have

$$\begin{aligned}
 (3.13) \quad &\left| \int_{|x-y|>\varepsilon} [k(x,y) - k(x_0,y)] f_2(y) dy \right| \\
 &\leq c d(Q) \int_{|x-y|>2d(Q)}
 \end{aligned}$$

$$\begin{aligned} & \frac{\left[1 + \log \frac{|x-y|}{d(Q)} \right] S_q(\nabla A)(x_1) + |\nabla A(y) - m_Q(\nabla A)|}{|x-y|^{n+1}} |f(y)| dy \\ & + c S_q(\nabla A)(x_1) d(Q) \int_{|x-y|>2d(Q)} \frac{|f(y)|}{|x-y|^{n+1}} dy \\ & + c d(Q) \int_{|x-y|>2d(Q)} \frac{|\nabla A(y) - m_Q(\nabla A)||f(y)|}{|x-y|^{n+1}} dy \end{aligned}$$

where x_1 is any point Q . The first integral comes from applying (2.14), (2.15), and Remark (2.17). The second integral comes from applying (2.16). The third integral comes from (2.14).

All three integrals are bounded by $c S_q(\nabla A)(x_1) \Lambda_p(x_1)$ for any $x_1 \in Q$. The estimate for the first integral comes from (2.7) and (2.12), the estimate for the second integral comes from (2.7), and the estimate for the third integral comes from using (2.12) again.

Finally, putting together all the estimates and choosing the point $z \in Q_j$, we have for $x \in Q_j$

$$\begin{aligned} (3.14) \quad |C_\varepsilon(\nabla A, f_2)(x)| & \leq \lambda + c S_q(\nabla A)(z) \Lambda_p(f)(z) \\ & \leq \lambda + c\gamma\lambda. \end{aligned}$$

The f_2 estimate: $\varepsilon > 1000n d(Q)$.

Let Q_ε be a cube with the same center as Q , with sides parallel to the axes and with diameter ε . We define

$$A_{Q_\varepsilon}(x) = A(x) - \sum_{j=1}^n c_j x_j \quad \text{where} \quad c_j = m_{Q_\varepsilon}(D_j A).$$

Let

$$k_\varepsilon(x, y) = \frac{A_{Q_\varepsilon}(x) - A_{Q_\varepsilon}(y) - \nabla A_{Q_\varepsilon}(y) \cdot (x - y)}{|x - y|^{n+1}} \Omega(x - y).$$

For $x \in Q_j$ and $\varepsilon > 1000n d(Q)$, $C_\varepsilon(\nabla A, f_2)(x) = C(\nabla A, f)(x)$. Then proceeding as in (3.10), for $x \in Q_j$, $x_0 \in Q \cap X^c$,

$$\begin{aligned} (3.15) \quad |C_\varepsilon(\nabla A, f_2)(x)| & \leq \left| \int_{|x-y|>\varepsilon} [k_\varepsilon(x, y) - k_\varepsilon(x_0, y)] f(y) dy \right| \\ & + \int_{\varepsilon < |x-y| < 5\varepsilon} |k_\varepsilon(x, y) f(y)| dy \\ & + \int_{\varepsilon < |x_0-y| < 5\varepsilon} |k_\varepsilon(x_0, y) f(y)| dy \end{aligned}$$

$$+ \left| \int_{|x_0-y|>\varepsilon} k_\varepsilon(x_0, y) f(y) dy \right| \\ \leq \lambda + c\gamma\lambda.$$

The second and third integrals satisfy the same estimate as in (3.10) because $\varepsilon = d(Q_\varepsilon)$. The last integral is less than or equal to λ since $x_0 \notin X$.

To estimate the first integral we proceed as in (3.12) with Q_ε replacing Q . The argument requires some modification since $|x - x_0|$ is no longer of the same magnitude as $d(Q_\varepsilon)$. Choose a point x_ε where $|x - x_\varepsilon| = 2\varepsilon$. Then $\varepsilon < |x_\varepsilon - x_0| < 3\varepsilon$. If we replace $A_{Q_\varepsilon}(x) - A_{Q_\varepsilon}(x_0)$ by $(A_{Q_\varepsilon}(x) - A_{Q_\varepsilon}(x_\varepsilon)) + (A_{Q_\varepsilon}(x_\varepsilon) - A_{Q_\varepsilon}(x_0))$ the rest of the argument holds and we have the estimate

$$(3.16) \quad \left| \int_{|x-y|>\varepsilon} [k_\varepsilon(x, y) - k_\varepsilon(x_0, y)] f(y) dy \right| \leq c\gamma\lambda.$$

Putting together all the f_2 estimates we get the pointwise estimate

$$(3.17) \quad C_*(\nabla A, f_2)(x) \leq \lambda + c\gamma\lambda$$

for $x \in Q_j$.

Conclusion. Choose $\gamma_0 < 1/c$ where c is the constant in (3.17). Assume $\gamma < \gamma_0$. Then $2\gamma - c\gamma\lambda > \lambda$ and so using estimate (3.9),

$$\begin{aligned} |x \in Q_j: C_*(\nabla A, f)(x) > 3\lambda, \quad S_q(\nabla A)(x)\Lambda_p(f)(x) \leq \gamma\lambda| \\ \leq |x \in Q_j: C_*(\nabla A, f_1)(x) > 2\lambda - c\gamma\lambda| + |x \in Q_j: C_*(\nabla A, f_2)(x) > \lambda + c\gamma\lambda| \\ \leq |x \in Q_j: C_*(\nabla A, f_1)(x) > \lambda| \\ \leq c\gamma^r |\bar{Q}| \\ \leq c'\gamma^r |Q_j|. \end{aligned}$$

Since the Q_j 's are disjoint we can sum over both sides of (3.17) and we get Theorem (1.1). Corollary (1.2) then follows by standard arguments.

REFERENCES

1. B. M. BAINSHANSKY & R. COIFMAN, *On singular integrals*, Proc. Sympos. Pure Math. Vol. 10, Amer. Math. Soc., Providence, R. I. (1967), 1-17.
2. A. P. CALDERÓN, *Commutators of singular integrals*, Proc. Nat. Acad. Sci. U.S.A. 53 (1965), 1092-1099.
3. C. P. CALDERÓN, *On commutators of singular integrals*, Studia Math. 53 (1975), 139-174.
4. R. COIFMAN & Y. MEYER, *Further results*, Seminar, Orsay, 1975.
5. R. COIFMAN, R. ROCHBERG & G. WEISS, *Factorization theorems for Hardy spaces in several variables*, Ann. of Math. 103 (1976), 611-635.
6. F. JOHN & L. NIRENBERG, *On functions of bounded mean oscillation*, Comm. Pure Appl. Math. 14 (1961), 415-426.
7. E. M. STEIN, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, NJ, 1970.

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